BSCB 694: Theory of Multivariate Statistics

Homework 1

Due: Thursday, Feb 28

1. Prove a Central Limit Theorem for S. Let $\mathbf{x}_i \sim (\mu, \Sigma)$ be iid p-vectors with finite fourth moments. Then

$$(n-1)^{-1/2}\text{vec}(S-\Sigma) \to N(0,\Omega)$$

in distribution for some Ω. Demonstrate that Ω is singular and find an alternative CLT which yields a non-singular limiting covariance.

2. A distribution $f(\mathbf{x})$ is spherically symmetric if the contours of f are hyper-spheres. That is: the set of $\mathbf{x}$ such that $f(\mathbf{x}) = c$ are all equi-distant from the origin. Show that the family $N(0, \sigma^2 I)$ are the only spherically symmetric distributions for which the components of $\mathbf{x}$ are independent.

3. Let X be a data matrix of size $n \times p$ from $N(0, \Sigma)$.

   (a) Demonstrate that

$$\text{vec}(AX) \sim N\left(0, \Sigma \otimes (AA^T)\right)$$

   (b) Show that AXB and CXD are independent if and only if either (a) $B^T\Sigma D = 0$ or (b) $AC^T = 0$.

   (c) For C symmetric, show that X^TCX and AXB are independent if either $B^T\Sigma = 0$ or $AC = 0$.

4. Demonstrate that if $M \sim W_p(\sigma, m)$ and $m \geq p$, then $|M|$ is $|\Sigma|$ times the product of p independent χ^2 rnaom variables with degrees of freedom m, $m-1, \ldots, m-p+1$. Hence conclude that $|M|$ is positive definite with probability 1.

   Hint: Proceed by induction. Show that $|M| = |M_{11}||M_{22.1}|$ with a similar decomposition for the determinant of Σ.

5. (a) Suppose $X_{n\times p}$ is a data matrix from $N(\mu, \Sigma)$. Show that the joint maximum likelihood estimates for μ and Σ under the constraint that $R\mu = r$ are

$$\hat{\mu} = \bar{\mathbf{x}} - SR^T(RSR^T)^{-1}(R\bar{\mathbf{x}} - \mathbf{r})$$

   and

$$\hat{\Sigma} = S + \mathbf{d}\mathbf{d}^T, \;\; \mathbf{d} = SR^T(RSR^T)^{-1}(R\bar{\mathbf{x}} - \mathbf{r})$$

   (b) Assume that Σ is known. Find the mean and variance of $\hat{\mu}$. Show that the variance of $\hat{\mu}$ is less than the variance of $\bar{\mathbf{x}}$ by a non-negative definite matrix.